Rec 3:

We want to find a Lagrangian that gives the Dirac equation:

$$\left(i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\nabla - \beta m\right)\psi = 0$$

where $\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}$ $\quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \left(\begin{matrix} \alpha^{i^2} = 1 \\ \beta^2 = 1 \end{matrix}\right)$

multiply by β:

$$\left(i\beta\frac{\partial}{\partial t} + i\beta\vec{\alpha}\cdot\nabla - m\right)\psi = 0$$

introduce Dirac matrices

$$\gamma^0 = \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \gamma^i = \beta\alpha^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}$$

$$\Rightarrow \left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\psi = 0$$

or $\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$

Feynman slash notation:

for any 4-vector A^μ $\quad \not{A} \equiv \gamma^\mu A_\mu$

$$\Rightarrow \boxed{(i\not{\partial} - m)\psi = 0}$$

Lagrangian for Dirac equation:

Now we want a Lorentz scalar Lagrangian that gives the Dirac equation.

In class you showed that $\psi^\dagger \gamma^0 \psi$ (not $\psi^\dagger \psi$) is a Lorentz scalar. $\bar{\psi} \equiv \psi^\dagger \gamma^0$

Try $\boxed{\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi}$

$$\frac{\partial \mathcal{L}}{\partial_\mu \bar{\psi}} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\not{\partial} - m)\psi = 0$$

$$\frac{\partial \mathcal{L}}{\partial_\mu \psi} = i\bar{\psi}\gamma^\mu \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu \psi} = i\partial_\mu \bar{\psi}\gamma^\mu$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi} \quad \Rightarrow \quad i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$

Symmetries of the Dirac Lagrangian and the "necessity" of the photon!

The Lagrangian is invariant under a <u>global</u> $U(1)$ transformation

$$\psi \to e^{i\alpha}\psi \qquad \Rightarrow \mathcal{L} \to i(e^{-i\alpha}\bar{\psi})\not{\partial}(e^{i\alpha}\psi) - m(e^{-i\alpha}\bar{\psi})(e^{i\alpha}\psi)$$
$$\bar{\psi} \to e^{-i\alpha}\bar{\psi} \qquad = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi = \mathcal{L}$$

Now let's try a <u>local</u> $U(1)$ transformation

$$\psi \to e^{ie\Gamma(x)}\psi$$
$$\bar{\psi} \to e^{-ie\Gamma(x)}\bar{\psi}$$

This time

$$\mathcal{L} \to i(e^{-ie\Gamma(x)}\bar{\psi})\not{\partial}(e^{ie\Gamma(x)}\psi) - m(e^{-ie\Gamma(x)}\bar{\psi})(e^{ie\Gamma(x)}\psi)$$

$$= i\, e^{-ie\Gamma(x)}\bar{\psi}\, e^{ie\Gamma(x)}\left[\not{\partial}\psi + ie(\not{\partial}\Gamma)\psi\right] - m\bar{\psi}\psi$$

$$= i\bar{\psi}\not{\partial}\psi - e\bar{\psi}\not{\partial}\Gamma\psi - m\bar{\psi}\psi = \mathcal{L} - e\bar{\psi}\not{\partial}\Gamma\psi$$

So this is not a symmetry of the theory.

Let's see what happens if we insist that this local $U(1)$ transformation is a symmetry of the theory. We need to add something to cancel the $e\bar{\Psi}\not\partial\Gamma\Psi$ term from the transformation.

Introduce a new vector field A^μ:

$$\mathcal{L} = i\bar{\Psi}\not\partial\Psi - m\bar{\Psi}\Psi - e\bar{\Psi}\not{A}\Psi \quad (\ast\ast)$$

If under the transformation,

$$\boxed{\begin{aligned} \Psi &\to \epsilon^{ie\Gamma(x)}\Psi \\ \bar{\Psi} &\to \epsilon^{-ie\Gamma(x)}\bar{\Psi} \\ A_\mu &\to A_\mu - \partial_\mu\Gamma \end{aligned}} \quad (\ast)$$

then

$$\mathcal{L} \to i\bar{\Psi}\not\partial\Psi - m\bar{\Psi}\Psi - e\bar{\Psi}\not\partial\Gamma\Psi - e\bar{\Psi}\not{A}\Psi + e\bar{\Psi}\not\partial\Gamma\Psi = \mathcal{L}$$

cancel

So we have succeeded!

Note that the local $U(1)$ transformation $(\circledast)$ is a gauge transformation:

$$A^0 \to A^0 - \frac{\partial \Gamma}{\partial t} \quad \text{and} \quad \vec{A} \to \vec{A} + \nabla \Gamma$$

is a symmetry just like in classical E&M if we call $A^0 = V$ the scalar and $\vec{A}$ the vector potentials.

Now we added a new field A_μ to make the local $U(1)$ a symmetry. What else can we do with it now that we are stuck with it? We want a Lagrangian that is a Lorentz scalar and invariant under under the transformations $(\circledast)$.

Let's try adding $A^\mu A_\mu$ to $\mathcal{L}$. Under $(\circledast)$:

$$A^\mu A_\mu \to A^\mu A_\mu - A^\mu \partial_\mu \Gamma - \partial^\mu \Gamma A_\mu - \partial^\mu \Gamma \partial_\mu \Gamma \neq A^\mu A_\mu$$

So this term is not allowed. If we could have it, adding $-\frac{1}{2} m^2 A^\mu A_\mu$ would give whatever particle A_μ is describing a mass m and so we see that whatever A_μ is it is <u>massless</u>.

Now try $\partial^\mu A_\mu$. Under (*):

$$\partial^\mu A_\mu \to \partial^\mu A_\mu - \partial^\mu \partial_\mu \Gamma \neq \partial^\mu A_\mu$$

So this term is not allowed.

How about terms with two derivatives?

There are three ways to make a Lorentz scalar with two derivatives:

$$\mathcal{L} = a\,\partial^\mu A^\nu \partial_\mu A_\nu + b\,\partial^\mu A^\nu \partial_\nu A_\mu + c\,\partial^\mu A_\mu \partial^\nu A_\nu$$

We already know the last one won't work $\Rightarrow c = 0$.

Rewrite remaining terms:

$$\mathcal{L} = a'\,\partial^\mu A^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + b'\,\partial^\mu A^\nu (\partial_\mu A_\nu + \partial_\nu A_\mu)$$

$$= (a' + b')\,\partial^\mu A^\nu \partial_\mu A_\nu + (b' - a')\,\partial^\mu A^\nu \partial_\nu A_\mu$$

$\Rightarrow a = a' + b'$ & $b = b' - a'$

or $a' = \frac{a-b}{2}$ & $b' = \frac{a+b}{2}$

Now check transformations under (*):

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu \partial_\nu \Gamma + \partial_\nu \partial_\mu \Gamma = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu A_\nu + \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\nu A_\mu - \partial_\mu \partial_\nu \Gamma - \partial_\nu \partial_\mu \Gamma \neq \partial_\mu A_\nu + \partial_\nu A_\mu$$

Since $\partial^\mu A^\nu \to \partial^\mu A^\nu - \partial^\mu \partial^\nu \Gamma$

we need $b' = 0$.

Define $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\Rightarrow \tilde{\mathcal{L}} = a' \partial^\mu A^\nu F_{\mu\nu} = \frac{a'}{2} F^{\mu\nu} F_{\mu\nu}$$

So we can add $\tilde{\mathcal{L}}$ to (⊗). a' is arbitrary

So choose $a' = -\frac{1}{2}$;

$$\boxed{\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi - e\bar{\Psi}\not{A}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}}$$

Can also be written in terms of the gauge covariant derivative

$D_\mu \equiv \partial_\mu + ieA_\mu$:

$$\boxed{\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}}$$

Call $J^\mu = e\bar{\Psi}\gamma^\mu\Psi$

$$\Rightarrow \mathcal{L} = i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi \underbrace{- J^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{Lagrangian for E\&M!}}$$

So A^μ describes the photon

So by being stubbarn about the local $U(1)$ symmetry we derived all of electromagnetism!

Varying wrt $\bar{\psi}$ now gives Dirac eq. for electron in EM field:

$$(i\not\partial - m - e\not A)\psi = 0.$$

The generalization of this to several fields $\psi_1, \dots, \psi_N$ is called <u>Yang-Mills</u> theory.

Here instead of (*)

$$\begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix} \to \underset{\substack{\uparrow \\ N \times N \text{ matrix}}}{U} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

For $N=2$ and being stubborn about a local $SU(2)$ transformation requires the $W^{\pm}$ and Z^0 bosons and the weak force.*

For $N=3$ and being stubborn about a local $SU(3)$ transformation requires the eight gluons and the strong force.

- To be completely correct when we go beyond electrodynamics to the weak force and the SM the gauge group is

$$SU(2)_L \times U(1)_Y$$

left hyperchange

The $U(1)_{em}$ and the physical photon field

electromagnetism

is a linear combination of the $SU(2)_L$ fields and the $U(1)_Y$ field as is the physical Z^0 field.

The $W^\pm$ fields are combinations just of the original $SU(2)_L$ fields.